Three Letter Identities

Algebraic identities in three letters intrigue me. They are very Polyaesque in that they seem to provide just the first step beyond the obvious. They also suggest many manipulations and "problems." This is just a little list which I may grow over time just to see if I can find any principle for organization.

Example 1: Product of Cyclical Pair Sums

$$(u+v)(v+w)(u+w) = (u+v+w)(uv+vw+uw) - uvw$$

The RHS is $e_1e_2 - e_3$, which is a very natural homogeneous object of third degree. The factorization e_1e_2 of LHS + uvw shows that this sum is in two useful algebraic ideals. It seems like good practice (Euler, Gauss) to rewrite a symmetric polynomial in terms of the elementary symmetric polynomials. There is an algorithm for this that is not too cumbersome to do by hand, and, of course, there is a giant literature that gets too general too fast to be entertaining. (source: CN page 163)

Example 2: Newton Girard

$$u^2+v^2+w^2 = (u+v+w)^2 - 2(uv+vw+uw) = e_1^2 - 2e_2$$

$$u^3+v^3+w^3 = (u+v+w)^3 - 3(u+v+w)(uv+vw+uw) + 3uvw = e_1^3 - 3e_1e_2 + 3e_3$$

Being pure formulas in e_1, e_2, and e_3 "they" hold in n variables for all $n \geq 3$.

Example 3: Vieta Argument and Cube Sum Factorization

$$P(x) = (x-u)(x-v)(x-w) = x^3 - e_1x^2 + e_2x - e_3 \quad \text{has roots } u, v, w.$$

If you sum, $P(u), P(v)$ and $P(w)$ you get

$$0 = u^3 + v^3 + w^3 - e_1(u^2+v^2+w^2) + e_2e_1 - 3e_3$$

which suggests an amusing factorization:

$$u^3+v^3+w^3 - 3uvw = (u+v+w)(uv+vw+uv-u^2-v^2-w^2)$$

This formula is simplest when you are given that $u+v+w=0$, so you have a relation between the products and the cubes. Some Olympiad problems have exploited this specific relationship.

Example 4: Old News

$$(u+v+w)^2 = u^2+v^2+w^2+2(uv+vw+uw) \qquad (1)$$

Well, duh. Still, you can't leave it out. It's $e_1^2 = s_2 + 2e_2$, and obviously it holds for any number of letters.

Further Topics for Exploration

1. Power sum representations in terms of symmetric polynomials
2. Vieta formulas and applications
3. Algorithms for writing polynomials in terms of the elementary symmetric polynomials
4. Identities from determinants, traces, matrix factorizations
5. Identities that come from iterations (Pell equations, etc.)
6. Identities from algorithms — e.g. Hunt-Dyson and Sptizer identities.
7. Which three letter identities automatically generalize to give one n-letter identities?
8. Which identities are only for three letters — no n letter analogs.
9. Homogenization: What's it all about? How does it help?

J. Michael Steele
http://www-stat.wharton.upenn.edu/∼steele/

Keywords: algebraic identities, three letter identities, mathematical olympiad,